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Osserman and conformally Osserman manifolds with warped and twisted product structure

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Abstract

We characterize Osserman and conformally Osserman Riemannian manifolds with the local structure of a warped product. By means of this approach we analyze the twisted product structure and obtain, as a consequence, that the only Osserman manifolds which can be written as a twisted product are those of constant curvature.

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1 Introduction

The group of isometries of any Riemannian two-point-homogeneous space acts transitively on the unit sphere bundle. This implies that any Riemannian two-point-homogeneous manifold is Osserman, i.e., the eigenvalues of the Jacobi operator are constant on the unit sphere bundle. Moreover, the converse was conjectured by Osserman and proved to be true in dimension different from 16 [9, 12, 13]. Later, the Osserman concept was extended to the conformal setting by analyzing the conformal Jacobi operator. Remarkably both conditions are equivalent for Einstein manifolds of dimension different from 4, 16. We refer to [3, 4, 5, 7] for basic results on conformally Osserman manifolds.

Warped products were introduced in [2] as a tool to construct Riemannian manifolds with non positive curvature. Furthermore, the fact that many cosmological models are warped products reinforces their importance in Lorentzian signature. Twisted products were introduced in [8] as a more general structure, since they include a wider variety of metrics than warped products. Both, warped and twisted products, are the underlying structure of many geometrical situations and they play an important role in conformal geometry.

In this paper we study the relation between these geometric properties and the algebraic structure of the metric tensor. More specifically we study which

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Osserman manifolds are warped or twisted products. Our approach relies on the fact that the metric is positive definite, and thus the results we obtain are in Riemannian signature. As an exception, the analysis we carry out in Section 3 does not require any restriction on the signature, and thus we show that a necessary condition for any four-dimensional warped product to be conformally Osserman is the local conformal flatness. In Section 4 we study Riemannian conformally Osserman warped products in arbitrary dimension, obtaining the following result.

Theorem 1.1 *A Riemannian warped product $B \times_f F$ is conformally Osserman if and only if it is locally conformally flat.*

Finally, we also give in Section 4 a complete characterization of Osserman twisted products for positive definite metrics.

Theorem 1.2 *A Riemannian twisted product $B \times_f F$ is pointwise Osserman if and only if it is a space of constant sectional curvature.*

2 Preliminaries

Let (M, g) be a Riemannian manifold of dimension n. Let ∇ denote the Levi-Civita connection and R the curvature tensor chosen with the sign convention given by $R(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$. For any orthonormal basis $\{E_1, \ldots, E_n\}$, the Ricci tensor and the scalar curvature are defined as follows:

$$\rho(X,Y) = \sum_{i=1}^{n} R(X, E_i, Y, E_i)\,, \qquad \tau = \sum_{i=1}^{n} \rho(E_i, E_i)\,.$$

The curvature tensor R decomposes as $R = C \bullet g + W$, where C is the Schouten tensor, W is the Weyl tensor and $\bullet$ denotes de Kulkarni–Nomizu product. Then let

$$\mathcal{J}(X)Y = R(X,Y)X, \quad \text{and} \quad \mathcal{J}_W(X)Y = W(X,Y)X$$

be the usual *Jacobi operator* and the *conformal Jacobi operator*, respectively. (M, g) is said to be *pointwise Osserman* (respectively, *conformally Osserman*) if for every $p \in M$ the Jacobi operator $\mathcal{J}_p$ (respectively, the conformal Jacobi operator) has constant eigenvalues on the unit tangent sphere $S_p = \{x \in T_pM : g_p(x,x) = 1\}$, where T_pM is the space tangent to M on p. The relation between these two concepts was pointed out in [5], showing that (M, g) *is pointwise Osserman if and only if* (M, g) *is Einstein and conformally Osserman.*

Warped product structures

Let (B, g_B) and (F, g_F) be pseudo-Riemannian manifolds and let $f : B \longrightarrow \mathbb{R}^+$ be a positive function on B. The product manifold $M = B \times F$ endowed with the metric

$$g = g_B \oplus f^2 g_F$$

is called the *warped product* $B \times_f F$ with *base* B, *fiber* F and *warping function* f. The geometry of a warped product is described in terms of the curvature tensor as follows.

Lemma 2.1 [2] *Let* $(M, g) = B \times_f F$ *be a warped product. Let* $X, Y, Z \in \mathfrak{X}(B)$ *and let* $U, V, W \in \mathfrak{X}(F)$*. The curvature tensor* R *is given by:*

(i) $R(X, Y)Z$ *is the lift of* $R^B(X, Y)Z$ *on* B*,*

(ii) $R(U, X)Y = \frac{H_f(X,Y)}{f} U$*,*

(iii) $R(X, Y)U = R(U, V)X = 0$*,*

(iv) $R(X, U)V = \frac{\langle U,V \rangle}{f} \nabla_X(\nabla f)$*,*

(v) $R(U, V)W = R^F(U, V)W - \frac{\langle \nabla f, \nabla f \rangle}{f^2} (\langle U, W\rangle V - \langle V, W\rangle U)$*.*

Twisted product structures

Let (B, g_B) and (F, g_F) be pseudo-Riemannian manifolds and let $f : B \times F \longrightarrow \mathbb{R}^+$ be a positive function on $B \times F$. The product manifold $M = B \times F$ endowed with the metric

$$g = g_B \oplus f^2 g_F$$

is called the *twisted product* $B \times_f F$. The same terminology of *base* and *fiber* applies in this case, whereas f is referred to as the *twisting function*. In the following lemma the geometry of a twisted product is described. For simplicity in some of the expressions we use $\xi = Log\, f$ instead of f.

Lemma 2.2 [10], [14] *Let* $(M, g) = B \times_f F$ *be a twisted product. For* $X, Y, Z \in \mathfrak{X}(B)$ *and* $U, V, W \in \mathfrak{X}(F)$ *the curvature tensor* R *is given by:*

(i) $R(X, Y)Z = R^B(X, Y)Z$*,*

(ii) $R(U, X)Y = (H_\xi(X, Y) + X(\xi)Y(\xi))U$*,*

(iii) $R(X, U)V = g(U, V)(X(\xi)\nabla\xi + h_\xi(X)) - XV(\xi)U$*,*

(iv) $R(U, V)X = XV(\xi)U - XU(\xi)V$*,*

(v)
$$\begin{aligned} R(U, V)W = R^F(U, V)W \quad &+ \quad g(\nabla\xi, \nabla\xi)(g(U, W)V - g(V, W)U) \\ &+ \quad H_\xi(V, W)U - H_\xi(U, W)V \\ &+ \quad g(V, W)h_\xi(U) - g(U, W)h_\xi(V) \\ &+ \quad W(\xi)(V(\xi)U - U(\xi)V) \\ &+ \quad (U(\xi)g(V, W) - V(\xi)g(U, W))\nabla\xi\,. \end{aligned}$$

Remark 2.3 The difference between warped and twisted products is pointed out by reducing the base B to a single point $\{p\}$. Then, warped products $\{p\} \times_f F$ become homothetic metrics on the fiber F, while twisted products $\{p\} \times_f F$ are in one to one correspondence with conformal metrics on F.

3 Four-dimensional setting

Four-dimensional geometry is of special interest in the context of Osserman and conformally Osserman geometries. Three-dimensional Osserman metrics are of constant curvature and moreover, the Weyl tensor has no meaning in dimension three. Therefore, dimension four is the first non-trivial case to be considered.

A proper feature of dimension four is that the Hodge star operator is idempotent and the curvature tensor further decomposes as $R = C \bullet g + W_+ + W_-$, where $W_\pm$ denote the self-dual and anti-self-dual components of the Weyl tensor. Moreover, since *a four-dimensional pseudo-Riemannian manifold is conformally Osserman if and only if it is (anti-)self-dual* (see [4] and [7] for a proof of this fact), our approach in the present section relies on the study of the self-dual and the anti-self-dual Weyl curvature operators.

3.1 Four-dimensional warped products

The following lemma, which relates self-duality and anti-self-duality, is a previous step to the characterization of conformally Osserman warped products in dimension four.

Lemma 3.1 *A four-dimensional product manifold $B \times F$ is self-dual if and only if it is anti-self-dual.*

Proof. Since any Lorentzian four-manifold is self-dual if and only if it is anti-self-dual, next we consider Riemannian $(++++)$ or neutral signature $(++--)$ direct products $B \times F$. Let $p \in B \times F$ be an arbitrary point. Let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal basis of the tangent space and let $\{e^1, e^2, e^3, e^4\}$ be the corresponding dual basis. Consider the following orthonormal basis for the self-dual and anti-self-dual spaces:

$$\Lambda^\pm = \mathrm{Span}\,\{E_1^\pm = (e^1 \wedge e^2 \pm \epsilon_3\epsilon_4 e^3 \wedge e^4)/\sqrt{2}, E_2^\pm = (e^1 \wedge e^3 \mp \epsilon_2\epsilon_4 e^2 \wedge e^4)/\sqrt{2}, \\ E_3^\pm = (e^1 \wedge e^4 \pm \epsilon_2\epsilon_3 e^2 \wedge e^3)/\sqrt{2}\},$$

where $\epsilon_i = g(e_i, e_i)$. The diagonal terms of the self-dual and anti-self-dual matrix associated to these basis are given by:

$$\begin{aligned}
2W_{11}^+ &= W_{1212} + W_{3434} + 2\epsilon_3\epsilon_4 W_{1234}, \\
2W_{11}^- &= W_{1212} + W_{3434} - 2\epsilon_3\epsilon_4 W_{1234}, \\
2W_{22}^+ &= W_{1313} + W_{2424} - 2\epsilon_2\epsilon_4 W_{1324}, \\
2W_{22}^- &= W_{1313} + W_{2424} + 2\epsilon_2\epsilon_4 W_{1324}, \\
2W_{33}^+ &= W_{1414} + W_{2323} + 2\epsilon_2\epsilon_3 W_{1423}, \\
2W_{33}^- &= W_{1414} + W_{2323} - 2\epsilon_2\epsilon_3 W_{1423},
\end{aligned} \tag{1}$$

but for a direct product we have $W_{1234} = W_{1324} = W_{1423} = 0$, so $W_{11}^+ = W_{11}^-$, $W_{22}^+ = W_{22}^-$ and $W_{33}^+ = W_{33}^-$. The remaining terms are:

$$\begin{array}{rcl} 2W_{12}^+ &=& W_{1213} - \epsilon_2\epsilon_4 W_{1224} + \epsilon_3\epsilon_4 W_{3413} - \epsilon_2\epsilon_3 W_{3424}, \\ 2W_{12}^- &=& W_{1213} + \epsilon_2\epsilon_4 W_{1224} - \epsilon_3\epsilon_4 W_{3413} - \epsilon_2\epsilon_3 W_{3424}, \\ 2W_{13}^+ &=& W_{1214} + \epsilon_2\epsilon_3 W_{1223} + \epsilon_3\epsilon_4 W_{3414} + \epsilon_2\epsilon_4 W_{3423}, \\ 2W_{13}^- &=& W_{1214} - \epsilon_2\epsilon_3 W_{1223} - \epsilon_3\epsilon_4 W_{3414} + \epsilon_2\epsilon_4 W_{3423}, \\ 2W_{23}^+ &=& W_{1314} + \epsilon_2\epsilon_3 W_{1323} - \epsilon_2\epsilon_4 W_{2414} - \epsilon_3\epsilon_4 W_{2423}, \\ 2W_{23}^- &=& W_{1314} - \epsilon_2\epsilon_3 W_{1323} + \epsilon_2\epsilon_4 W_{2414} - \epsilon_3\epsilon_4 W_{2423}. \end{array} \tag{2}$$

There are two different cases to be analyzed: $dim\, B = dim\, F = 2$ and $dim\, B = 1$, $dim\, F = 3$.

Suppose first that $dim\ B = 2$, $dim\ F = 2$. Let $e_1, e_2 \in \mathfrak{X}(B)$ and $e_3, e_4 \in \mathfrak{X}(F)$. Then

$$\begin{gathered} W_{1213} = W_{1224} = W_{3413} = W_{3424} = 0, \\ W_{1214} = W_{1223} = W_{3414} = W_{3423} = 0, \\ W_{1314} = -\tfrac{\epsilon_1}{2}\rho_{34}, W_{1323} = -\tfrac{\epsilon_3}{2}\rho_{12}, W_{2414} = -\tfrac{\epsilon_4}{2}\rho_{12}, W_{2423} = -\tfrac{\epsilon_2}{2}\rho_{34}. \end{gathered}$$

Hence $W_{12}^+ = W_{12}^- = W_{13}^+ = W_{13}^- = 0$. Now, since the signature is Riemannian or neutral, we have $\epsilon_1 = \epsilon_2\epsilon_3\epsilon_4$, therefore

$$W_{23}^+ = -\frac{1}{4}(\epsilon_1 - \epsilon_2\epsilon_3\epsilon_4)\rho_{34} = 0 = W_{23}^-.$$

Suppose now that $dim\ B = 3$ and $dim\ F = 1$. Let $e_1, e_2, e_3 \in \mathfrak{X}(B)$ and $e_4 \in \mathfrak{X}(F)$. Then

$$W_{1224} = W_{3413} = 0, \quad W_{1214} = W_{3423} = 0, \quad W_{1314} = W_{2423} = 0,$$

so $W_{12}^+ = W_{12}^-$, $W_{13}^+ = -W_{13}^-$ and $W_{23}^+ = -W_{23}^-$.

The relations between the self-dual and the anti-self-dual components show that in all cases the self-dual and anti-self-dual operators simultaneously vanish. Hence the result follows. □

The main result of this section reads as follows:

Theorem 3.2 *A four-dimensional pseudo-Riemannian warped product manifold is conformally Osserman if and only if it is locally conformally flat.*

Proof. If the signature is Lorentzian then the result follows from [5]. Hence assume hereafter that the signature is Riemannian or neutral. A warped product $B \times_f F$ is in the conformal class of $(B \times F, \frac{1}{f^2} g_B \oplus g_F)$, which is a direct product. Thus, since the two properties under consideration are conformal invariants, it suffices for our purpose to restrict our analysis to direct products. From Lemma 3.1 a direct product is self-dual if and only if it is anti-self-dual and the result follows. □

3.2 Four-dimensional twisted products

Note that the relations given in the proof of Lemma 3.1 between the self-dual and anti-self-dual components still hold if we consider a twisted product $B \times_f F$ with $dim\ F = 1$. Hence Lemma 3.1 also holds in this particular case. Moreover, for a twisted product $B \times_f F$ with $dim\ B = 1$ we consider a conformal change by $\frac{1}{f^2}$ to get $(B \times F, \frac{1}{f^2} g_B \oplus g_F)$, so that the dimension of the fiber is 1. Again the result holds since the conditions under consideration are conformally invariant, thus we obtain the following result.

Theorem 3.3 *Let $B \times_f F$ be a four-dimensional pseudo-Riemannian twisted product with $\dim B = 1$ or $\dim B = 3$. Then (M, g) is conformally Osserman if and only if it is locally conformally flat.*

In view of Theorem 3.3 one may wonder if this result holds in general for a four-dimensional twisted product. This is not the case, as next example shows.

Example 3.4 Consider the twisted product $(M, g) = \mathbb{R}^2 \times_f \mathbb{R}^2$, with twisting function $f(x_1, x_2, x_3, x_4) = e^{x_1 x_3 - x_2 x_4}$. A lengthy but straightforward calculation shows that

$$W^+ = \begin{pmatrix} 0 & 0 & \frac{1}{2}(1 + e^{x_1 x_3 - x_2 x_4}) \\ 0 & 0 & 0 \\ \frac{1}{2}(1 + e^{x_1 x_3 - x_2 x_4}) & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad W^- = 0.$$

Hence (M, g) is self-dual but it is not anti-self-dual and, consequently, it is conformally Osserman but not locally conformally flat.

Although we have just given a general negative answer to the extension of Theorem 3.2 to twisted products, the following result shows the non-existence of compact (anti-)self-dual twisted products which are not locally conformally flat if the metric is positive definite.

Theorem 3.5 *Let $(M, g) = B \times_f F$ be a compact Riemannian twisted product such that $\dim\ B, F = 2$. Then $B \times_f F$ is conformally Osserman if and only if it is locally conformally flat. Moreover, in such a case it is actually a warped product.*

Proof. Since $B \times_f F$ is conformally Osserman, it is self-dual or anti-self-dual. Suppose $B \times_f F$ is self-dual (reversing the orientation interchanges the roles of the self-dual and the anti-self-dual components). Since B and F are 2-dimensional and oriented, let J^B and J^F be orthogonal complex structures on B and F, respectively. Then $B \times_f F$ is Hermitian and opposite Hermitian just considering the complex structure $J^B \oplus \pm J^F$. If $B \times_f F$ is self-dual, then results in [1] show that $B \times_f F$ is locally conformally flat or conformally equivalent to $\mathbb{CP}^2$ with the Fubini-Study metric or to a compact quotient of the unit ball in $\mathbb{C}^2$ with the Bergman metric. Hence, it follows that $B \times_f F$ is locally conformally

flat or a locally conformally Kähler manifold. Assume J is locally conformally Kähler. Let X be a vector on the base and U a vector on the fiber. Hence $\{X, JX, U, JU\}$ is an adapted basis for the product. Since $(B \times_f F, J)$ is locally conformally Kähler, then there exists ψ such that $(B \times F, \psi\, g^B \oplus \psi\, f^2 g^F)$ is Kähler on a suitable open set (note that this is a doubly twisted product). Then

$$\begin{aligned}
(\nabla_X J)U &= \nabla_X(JU) - J\nabla_X U \\
&= JU(\ln\psi)X + X(\ln\psi\, f^2)JU - U(\ln\psi)JX - X(\ln\psi\, f^2)JU \\
&= JU(\ln\psi)X - U(\ln\psi)JX,
\end{aligned}$$

$$\begin{aligned}
(\nabla_U J)X &= \nabla_U(JX) - J\nabla_U X \\
&= JX(\ln\psi\, f^2)U + U(\ln\psi)JX - X(\ln\psi\, f^2)JU - U(\ln\psi)JX \\
&= JX(\ln\psi\, f^2)U - X(\ln\psi\, f^2)JU,
\end{aligned}$$

from where $U(\ln\psi) = 0$, $JU(\ln\psi) = 0$, $X(\ln\psi f^2) = 0$ and $JX(\ln\psi f^2) = 0$. This implies that ψ is constant over F and one proceeds in an analogous way to show that ψf^2 is constant over B too. Therefore f decomposes as a product $f = f_B f_F$ so that f_B is constant on F and f_F is constant on B, hence $B \times_f F$ is indeed a warped product and by Theorem 3.2 it is locally conformally flat. $\square$

4 Higher dimensional setting

Motivated by the results of previous section, one may wonder if an arbitrary conformally Osserman warped product is locally conformally flat. It has been shown in [5] that any conformally Osserman Lorentzian manifold is locally conformally flat but the following is a counterexample to this fact in arbitrary indefinite signature (not Lorentzian).

Example 4.1 Consider the manifold $\mathcal{B} \times \mathbb{R}^k_\nu$, where $\mathcal{B} = (\mathbb{R}^4, g_{\mathcal{B}})$ and g_B is given by

$$g_{\mathcal{B}}(x_1, x_2, x_3, x_4) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & c(x_3, x_4) \\ 0 & 1 & c(x_3, x_4) & 0 \end{pmatrix},$$

where (x_1, x_2, x_3, x_4) are coordinates in $\mathcal{B}$. Note that $\mathcal{B} = (\mathbb{R}^4, g_{\mathcal{B}})$ is a strictly Walker metric with signature $(2, 2)$ and $\mathbb{R}^k_\nu$ is endowed with an Euclidian metric of signature $(\nu, k - \nu)$. Then the only non-null component of the curvature, up to the usual symmetries, is

$$R(\partial_3, \partial_4, \partial_3, \partial_4) = \partial_3\partial_4 c(x_3, x_4).$$

A straightforward calculation shows that M is Ricci flat and thus the only non-null component of the Weyl tensor, up to the usual symmetries, is

$$W(\partial_3, \partial_4, \partial_3, \partial_4) = \partial_3 \partial_4 c(x_3, x_4).$$

Then $\mathcal{B} \times \mathbb{R}^k_\nu$ is conformally Osserman with nilpotent conformal Jacobi operator; however it is not locally conformally flat unless $\partial_3 \partial_4 c(x_3, x_4) = 0$.

In what follows we show that the result in theorems 3.2 and 3.3 still holds in the Riemannian setting for arbitrary dimension. We proceed in an analogous way to the previous section and, as a preliminary step, we characterize conformally Osserman direct products.

Lemma 4.2 *Let (M, g) be a conformally Osserman Riemannian manifold which decomposes as a direct product $B \times F$. Then $W^M = 0$.*

Proof. Let $p \in M$ be an arbitrary point. To establish notation, set $b = dim\, B$ and $d = dim\, F$ and use superindexes B and F to refer to manifolds B and F, respectively. On T_pM we can choose an orthonormal basis $\{e_1, \ldots, e_b, f_1, \ldots, f_d\}$, with $\{e_i\} \subset T_p^B M$ and $\{f_i\} \subset T_p^F M$, which diagonalizes the Ricci tensor.

Note that $\mathcal{J}_W(e_i)e_j, \mathcal{J}_W(f_i)e_j \in T_p^B M$ and $\mathcal{J}_W(f_i)f_j, \mathcal{J}_W(e_i)f_j \in T_p^F M$. Indeed, from the expressions of the curvature in Lemma 2.1 we compute:

$$\begin{aligned} \mathcal{J}_W(e_i)e_j &= \mathcal{J}_R(e_i)e_j - \tfrac{1}{n-2}\left(\rho(e_i, e_i) + \rho(e_j, e_j) - \tfrac{\tau}{n-1}\right) e_j, \text{ and} \\ \mathcal{J}_W(e_i)f_j &= -\tfrac{1}{n-2}\left(\rho(e_i, e_i) + \rho(f_j, f_j) - \tfrac{\tau}{n-1}\right) f_j. \end{aligned} \tag{3}$$

Therefore, f_j is an eigenvector for every $\mathcal{J}_W(e_i)$ (analogously, e_i is an eigenvector for every $\mathcal{J}_W(f_j)$). Also notice that mixed terms of the Weyl tensor vanish, that is, $W(e_i, f_j)e_k = 0$ and $W(f_i, e_j)f_k = 0$ if $i \neq k$.

Suppose $\mathcal{J}_W(e_1)f_j = \lambda f_j$, $\mathcal{J}_W(e_1)f_k = \mu f_k$ with $j \neq k$. Recall the Rakić duality principle [15]: *"for an Osserman Riemannian manifold $\mathcal{J}_A(X)Y = \lambda Y$ if and only if $\mathcal{J}_A(Y)X = \lambda X$"*. This principle also applies for conformally Osserman Riemannian manifolds (indeed it is true in a purely algebraic context for any Osserman algebraic curvature tensor, see [11]). Hence we compute

$$\begin{aligned} \mathcal{J}_W(\cos\theta f_j + \sin\theta f_k)e_1 &= \cos^2\theta \mathcal{J}_W(f_j)e_1 + \sin^2\theta \mathcal{J}_W(f_k)e_1 \\ &\quad + \cos\theta\sin\theta(W(f_j, e_1)f_k + W(f_k, e_1)f_j) \\ &= \cos^2\theta \mathcal{J}_W(f_j)e_1 + \sin^2\theta \mathcal{J}_W(f_k)e_1 \\ &= \left(\cos^2\theta\lambda + \sin^2\theta\mu\right) e_1. \end{aligned}$$

This shows that e_1 is an eigenvector for $\mathcal{J}_W(\cos\theta f_j + \sin\theta f_k)$ associated to the eigenvalue $\cos^2\theta\lambda + \sin^2\theta\mu$; but, since the eigenvalues are constant, we conclude that $\lambda = \mu$. By repeating this argument, we show that all the eigenvalues of $\mathcal{J}_W(e_1)$ associated to eigenvectors f_j in $T_p^F M$ are equal.

Next, we show that all the eigenvalues of $\mathcal{J}_W(e_1)$ are indeed equal. Take a unitary vector $x \in T_p^B M$ such that $\mathcal{J}_W(e_1)x = \nu x$. Then

$$\begin{aligned}
\mathcal{J}_W(\cos\theta e_1 + \sin\theta f_1)x &= \cos^2\theta \mathcal{J}_W(e_1)x + \sin^2\theta \mathcal{J}_W(f_1)x \\
&\quad + \cos\theta\sin\theta\,(W(e_1,x)f_1 + W(f_1,x)e_1) \\
&= \left(\cos^2\theta\nu + \sin^2\theta\lambda\right)x,
\end{aligned}$$

and necessarily $\lambda = \nu$. Since the trace of $\mathcal{J}_W(\cdot)$ is zero, we conclude that all the eigenvalues of $\mathcal{J}_W(\cdot)$ vanish and hence the Weyl tensor is zero. □

Proof of Theorem 1.1. It follows immediately from Lemma 4.2 just using that any warped product metric is in the conformal class of a product metric.□

4.1 Osserman condition on twisted products

It was shown in [6] that a locally conformally flat twisted product with factors of dimension greater than one may be expressed as a warped product. Hence it arises as a natural question whether a conformally Osserman twisted product can be reduced to a warped one. The answer to this question is positive for odd dimensions, since in this case the conformally Osserman condition is equivalent to local conformal flatness (see [5]). On the contrary conformally Osserman twisted products with base and fiber of dimension ≥ 2 do not necessarily reduce to warped products as Example 3.4 shows. However, we see in this section that manifolds with twisted product structure and fiber or base of dimension one behave quite differently.

Theorem 4.3 *A twisted product $B \times_f F$, with dim $B = 1$ or $\dim F = 1$, is conformally Osserman if and only if it is locally conformally flat.*

Proof. If $dim\ B = 1$, $B \times_f F$ is in the conformal class of $F \times_{1/f} B$ whose fiber is one-dimensional, so we may suppose without loss of generality that $dim\ F = 1$. Let $p \in M$ be an arbitrary point. Let v be a unit vector on the fiber F. Now consider $\mathcal{J}_W(v)$ and take a basis of orthonormal eigenvectors $\{e_1, \dots, e_{n-1}\}$ tangent to the base of the twisted product. Denote by $\{\lambda_1, \dots, \lambda_{n-1}, 0\}$ the eigenvalues associated to the orthonormal basis $\{e_1, \dots, e_{n-1}, v\}$. Note that for $i \neq j$,

$$W(e_i, v, e_j, v) = \langle \mathcal{J}_W(v)e_i, e_j\rangle = 0,$$

and, for any i, j, k with $i \neq j$, $j \neq k$, $k \neq i$,

$$\begin{aligned}
W(e_i, v, e_j, e_k) &= W(e_j, e_k, e_i, v) \\
&= R(e_j, e_k, e_i, v) - \tfrac{1}{n-2}\left(\rho(e_j, e_i)\langle e_k, v\rangle + \rho(e_k, v)\langle e_j, e_i\rangle\right. \\
&\qquad - \rho(e_j, v)\langle e_k, e_i\rangle - \rho(e_k, e_i)\langle e_j, v\rangle \\
&\qquad \left. - \tfrac{\tau}{n-1}\langle e_j, e_i\rangle\langle e_k, v\rangle - \langle e_j, v\rangle\langle e_i, e_k\rangle\right) \\
&= 0.
\end{aligned}$$

Also note that $W(e_i, v, e_j, e_i) = 0$. Hence $W(e_i, v)e_j = 0$ for $i \neq j$ and $\mathcal{J}_W(e_i)v = \lambda_i v$. Then

$$\begin{aligned}\mathcal{J}_W(\cos\theta e_i + \sin\theta e_j)v &= \cos^2\theta \mathcal{J}_W(e_i)v + \sin^2\theta \mathcal{J}_W(e_j)v \\ &\quad + \cos\theta\sin\theta(W(e_i, v)e_j + W(e_j, v)e_i) \\ &= (\cos^2\theta\lambda_i + \sin^2\theta\lambda_j)v.\end{aligned}$$

Since M is conformally Osserman, the eigenvalues of the conformal Jacobi operator are constant. Therefore $\lambda_i = \lambda_j$. But, since $\mathrm{tr}(\mathcal{J}_W(\cdot)) = 0$, all the eigenvalues are zero and thus the Weyl tensor vanishes. Hence M is locally conformally flat. $\square$

Finally we take advantage of Theorem 4.3 to show that every Osserman twisted product has constant sectional curvature in the Riemannian setting.

Proof of Theorem 1.2. Assume $B \times_f F$ is Osserman. Then it is Einstein and conformally Osserman [5]. On the one hand, if $dim\, F > 1$ then, by results in [10], $B \times_f F$ is indeed a warped product and, from Theorem 1.1, $B \times_f F$ is locally conformally flat. On the other hand, if $dim\, F = 1$ we conclude from Theorem 4.3 that $B \times_f F$ is locally conformally flat too. Finally, since $B \times_f F$ is Einstein and locally conformally flat, it has constant sectional curvature. $\square$

Since any two-point-homogeneous manifold is Osserman, the following result is an immediate consequence of Theorem 1.2.

Corollary 4.4 *A two-point homogeneous space can be decomposed as a twisted product if and only if it is of constant sectional curvature.*

References

[1] V. Apostolov, J. Davidov, O. Muškarov; Compact self-dual Hermitian surfaces, *Trans. Amer. Math. Soc.* **348**, n 8, (1996), 3051–3063.

[2] R. L. Bishop, and B. O'Neill; Manifolds of negative curvature, *Trans. Amer. Math. Soc.* **145** (1969), 1–49.

[3] N. Blažić, P. Gilkey; Conformally Osserman manifolds and conformally complex space forms, *Int. J. Geom. Methods Mod. Phys.* **1** (2004), 97–106.

[4] N. Blažić, P. Gilkey; Conformally Osserman manifolds and self-duality in Riemannian geometry, *Proceedings DGA Conf.*, Czech Republic 2004 (2005), 15–18.

[5] N. Blažić, P. Gilkey, S. Nikčević, U. Simon; The spectral geometry of the Weyl conformal tensor, *PDEs, submanifolds and affine differential geometry*, Banach Center Publ. **69**, Polish Acad. Sci., Warsaw, 2005, 195–203.

[6] M. Brozos-Vázquez, E. García-Río, R. Vázquez-Lorenzo; Some remarks on locally conformally flat static spacetimes, *J. Math. Phys.* **46** (2005), 11pp.

[7] M. Brozos-Vázquez, E. García–Río, R. Vázquez-Lorenzo; Conformal Osserman four-dimensional manifolds whose conformal Jacobi operators have complex eigenvalues, *Proc. Royal Society A* **462** (2006), 1425–1441.

[8] B.-Y. Chen; Geometry of Submanifolds and its Applications, *Science University of Tokyo*, 1981.

[9] Q.-S. Chi; Curvature characterization and classification of rank-one symmetric spaces, *Pacific J. Math.* **150** (1991), 31–42.

[10] M. Fernández López, E. García-Río, D. Kupeli, B. Ünal; A Curvature Condition for a Twisted Product to be a Warped Product, *Manuscripta Math.* **106** (2001), 213–217.

[11] P. Gilkey; *Geometric Properties of Natural Operators Defined by the Riemann Curvature Tensor*, World Scientific (2001).

[12] Y. Nikolayevsky; Osserman manifolds of dimension 8, *Manuscr. Math.* **115** (2004), 31–53.

[13] Y. Nikolayevsky; Osserman Conjecture in dimension $n \neq 8, 16$, *Math. Ann.* **331** (2005), 505–522.

[14] R. Ponge, H. Reckziegel; Twisted products in pseudo-Riemannian geometry, *Geom. Dedicata* **48** (1993), 15–25.

[15] Z. Rakić; On duality principle in Osserman manifolds, *Linear Algebra Appl.* **296** (1999), 183–189.

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